- Let $\mathbf{x}^*$ be an equilibrium point of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ i.e. $\mathbf{f}(\mathbf{x}^*) = 0$
  Let D be an open set surrounding $\mathbf{x}^*$ and let $V : D \to \mathbb{R}$ be a continuously differentiable function such that
  1. $V(\mathbf{x}^*) = 0$ and $V(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{x}^*$
  2. $\dot{V}(\mathbf{x}\ ) = \nabla V \cdot \mathbf{f}(\mathbf{x}) \leq 0$

  then the equilibrium point $\mathbf{x}^*$ is **stable**

RECALL THAT $x^* = 0$ IS STABLE IFF, FOR ALL $\varepsilon > 0$, $\exists\ \delta$ SUCH THAT

$$\|x(0)\| \leq \delta \Rightarrow \|x(t)\| \leq \varepsilon \quad \forall\ t > 0$$

USE ① + ② TO PROVE THAT $\delta > 0$ EXISTS FOR ALL $\varepsilon > 0$ $\longleftarrow$ $\forall\ \varepsilon \in (0, \varepsilon_0]$, $\varepsilon_0 > 0$

① $\alpha \|x\| \leq V(x) \leq \beta \|x\|$ FOR SOME $\alpha, \beta > 0$ $\forall\ x : \|x\| \leq \varepsilon_0$

FOLLOWS FROM : $V \in C^1$, $V(0) = 0$, $V(x) > 0\ \forall\ x \neq 0$

$$\therefore \begin{cases} \|x\| \leq \delta \Rightarrow V(x) \leq \beta\delta \\ V(x) \leq \alpha\varepsilon \Rightarrow \|x\| \leq \varepsilon \end{cases} \qquad (\delta, \varepsilon \leq \varepsilon_0)$$

② $\|x(0)\| \leq \delta \Rightarrow V(x(0)) \leq \beta\delta \Rightarrow V(x(t)) \leq \beta\delta \Rightarrow \|x(t)\| \leq \varepsilon$

FOLLOWS FROM $\dot{V} \leq 0$

IF $\beta\delta \leq \alpha\varepsilon$

e.g. $\delta = \frac{\alpha}{\beta} \min\{\varepsilon_0, \varepsilon\} > 0$

- Let $\mathbf{x}^*$ be an equilibrium point of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ i.e. $\mathbf{f}(\mathbf{x}^*) = 0$
  Let D be an open set surrounding $\mathbf{x}^*$ and let $V : D \to \mathbb{R}$ be a continuously differentiable function such that
  1. $V(\mathbf{x}^*) = 0$ and $V(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{x}^*$
  2. $\dot{V}(\mathbf{x}\ ) = \nabla V \cdot \mathbf{f}(\mathbf{x}) \leq 0$

  then the equilibrium point $\mathbf{x}^*$ is **stable**
- $\mathbf{x}^*$ is **asymptotically** stable if we also have
  3. $\dot{V}(\mathbf{x}) < 0$ for all $\mathbf{x} \neq \mathbf{x}^*$

PROVE ASYMPTOTIC STABILITY, i.e.

$$\|x(t)\| \to 0 \quad \forall\ t \to \infty \quad \text{IF } \|x(0)\| \leq R, \text{ FOR SOME } R > 0$$

BY CONTRADICTION USING ③ AND $V(x) \geq 0$

IF $\dot{V}(x) < 0$ $\forall\ x \neq 0$ AND $\|x(t)\| \geq r$ $\forall t$ FOR SOME $r > 0$

THEN $\dot{V}(x(t)) < -\gamma$ $\forall\ t$ FOR SOME $\gamma > 0$

BUT THIS IS NOT POSSIBLE BECAUSE $V(x(t)) \geq 0$ $\forall\ t$

- Let $\mathbf{x}^*$ be an equilibrium point of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ i.e. $\mathbf{f}(\mathbf{x}^*) = 0$
  Let D be an open set surrounding $\mathbf{x}^*$ and let $V : D \to \mathbb{R}$ be a continuously differentiable function such that
  1. $V(\mathbf{x}^*) = 0$ and $V(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{x}^*$
  2. $\dot{V}(\mathbf{x}\ ) = \nabla V \cdot \mathbf{f}(\mathbf{x}) \leq 0$

  then the equilibrium point $\mathbf{x}^*$ is **stable**
- $\mathbf{x}^*$ is **asymptotically** stable if we also have
  3. $\dot{V}(\mathbf{x}) < 0$ for all $\mathbf{x} \neq \mathbf{x}^*$
- $\mathbf{x}^*$ is **globally asymptotically** stable if we further have
  4. $\lim_{\|\mathbf{x}\| \to \infty} V(\mathbf{x}) = \infty$ and $D = \mathbb{R}^n$

IF $V(x) \to \infty$ WITH $x = \sigma\hat{x}$ AS $\sigma \to \infty$, FOR ALL $\hat{x}$ SUCH THAT $\|\hat{x}\| = 1$

THEN $\alpha\|x\| \leq V(x) \leq \beta\|x\|$ $\forall\ x$, FOR SOME $\alpha, \beta > 0$

$\therefore$ ARGUMENTS FROM ①, ②, ③ APPLY GLOBALLY $\forall\ x \in \mathbb{R}^n$